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Some Concepts of $\widetilde{H} - Abundant$ Semigroup in Group

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Abstract

Shum [9] in his work on systems of semigroups and its application in constructing generalized cryptogroups introduced the concepts of Green $relation$ on $\widetilde{H} - abundant$ semigroups by using the generalized strong semilattice of semigroups, he also used a generalization of a well-known result of fountain on super-abundant semigroup to show that $\widetilde{H} - cryptogroup$ is regular $\widetilde{H} - crytogroup$ if and only if it is an $\widetilde{HG} - strong\ semilattice$ of completely $\tilde{J} - simple$ semigroups. In this paper further explanation were made and proves of some lemmas, additional characteristics of the $\widetilde{H} - abundant$ semigroup is proved. Lastly the concepts of $\widetilde{H} - abundant$ semigroup in group with examples is given.

***Keywords**: Green ~relations, $\widetilde{H} - abundant$ semigroups, $\widetilde{H} - cryptogroup$, Cosets and Congruence*

Introduction

The aspect of regularity in semigroup theory is an important one, this can be described as core semigroup because groups are regular semigroups with a unique idempotent element. Many scholars have used them in the course of their research. A regular semigroup is said to be completely regular if S is a union of some of it's subgroup (2), it was also proved by Clifford that if S is a completely regular semigroup whose idempotent are central then, such semigroup can be expressed as a strong semilattice of groups. Guo-Shum (7) showed that a perfect abundant semigroup can be expressed as a strong semilattice of cancellative planks.

Green relations can be used in describing the regular semigroups, Pastign (9) was first to introduce the Green$*\ relation$ which can be regarded as the Green relation in some oversemigroup. The Green$*\ relation$ was first observed to be applicable in the study of abundant semigroup and in particular super abundant semigroup fountain (4) and (5).

For any element $a, b \in S$ where S is a semigroup Fountain [4] defined The extended green relation as follows:

$L^* = \{(a,b) \in S \times S : (\forall x, y \in S^1)ax = ay \Leftrightarrow bx = by\}$,

$R^* = \{(a,b) \in S \times S : (\forall x, y \in S^1)xa = ya \Leftrightarrow xb = yb\}$,

$H^* = L^* \cap R^*, D^* = L^* \vee R^*$.

Later on, El-Qallali further generalized the Green ∗relations to Green ∼relations [3] as follows:

$\tilde{L}$ = {(a, b) ∈ S × S : $(\forall e \in E(S))\ ae = a \Leftrightarrow be = b\}$,

$\tilde{R} = \{(a,b) \in S \times S : (\forall e \in E(S))\ ea = a \Leftrightarrow eb = b\}$,

$\tilde{H} = \tilde{L} \cap \tilde{R},\ \tilde{D}\, s = \tilde{L} \vee \tilde{R}$

It is obvious that $\tilde{L}$ and the $\tilde{R}$ are equivalent relation on S, however, the $\tilde{L}$ relation is not necessary to be right compatible with the semigroup multiplication and the R relation is not necessary to be left compatible with the semigroup multiplication. If a semigroup S is regular, then every L-class of S contains at least one idempotent, and so does every R-class of S. If S is a completely regular semigroup, then every H-class of S contains an idempotent. According to Fountain [4], a semigroup is abundant if every L^* and R^*class of S contains some idempotent. In other words, the term "abundant" means that the semigroup has plenty of idempotent. Clearly, we have L^* = L on the set of all regular elements of a semigroup. Thus, regular semigroups are obviously special abundant semigroups. Hence Fountain called a semigroup superabundant [4] if every of its H* -classes contains an idempotent. Obviously, completely regular semigroups are special superabundant semigroups. Following Elqallali [3], we call a semigroup S a semi-abundant semigroup if every $\tilde{L}$-class and every $\tilde{R}$-class of S contain at least one idempotent. A semigroup S is called $\tilde{H}$-abundant if every $\tilde{H}$-class contains an idempotent of S. Clearly, the $\tilde{H}$-abundant semigroups are generalizations of superabundant semigroups in the class of semiabundant semigroups. One can easily see that $\tilde{L}$ = L on the set of regular elements in any $\tilde{H}$-abundant semigroup.

Now we notice that from the various extension of the green relation that $D^* = L^* \vee R^* \neq R^* \circ L^*$, but $\tilde{D} = \tilde{L} \vee \tilde{R} = \tilde{R} \circ \tilde{L}$

Proof

$$\tilde{L} = \{(a,b) \in S \times S: (\forall e \in E(s))ae = a \Leftrightarrow be = b$$

$$\tilde{R} = \{(a,b) \in S \times S: (\forall e \in E(s))ea = a \Leftrightarrow eb = b$$

Let S be a semigroup, let $a, b \in \tilde{R} \circ \tilde{L}$ then there exists $C\ in\ S$

Such that $a\tilde{L}$c and $c\tilde{R}b$ and $c\tilde{L}$b so that

$ae = c \quad fc = b$

$ce = a \quad fb = c$

Were $e, f \in E(S)$

Then let $u = fce$

$$fa = fce = u$$

$$fu = ffce = fbe = ce = a$$

So that we have $a\tilde{R}u$ secondly

$$ue = fcee = fae = fc = b$$

$be = fce = u$ so that we will have $u\tilde{L}b$

Next is to show that $D^* = L^* \vee R^* \neq R^* \circ L^*$.

The $\tilde{L}$ relation is not necessary right compactible unlike l^* , we say that a relation ρ on S is right compactible if $(\forall s, t, a \in S)$ $(s,t) \in \rho \Longrightarrow (as, at) \in \rho$ so whenever $a \notin E(S)$, $\tilde{L}$ will not be right compatible this implies that $\tilde{R}$ is not always left compactible like wise.

Let denote the $\tilde{L} - class$ containing the element a of the semigroup S by $\widetilde{La}$ and observe that $L \sqsubseteq L^* \subseteq \tilde{L}$. For every L-relation there always exist a corresponding dual relation namely R-relation.

It is obvious that there exists at most one idempotent of the semigroup S in each class, now if $e \in \widetilde{Ha} \cap E(S)$ for some $a \in S$ then for any $x \in \widetilde{Ha}$, we clearly have that $x = ex = xe$

If every class of $\widetilde{H}$ has at most one idempotent element and of course $\forall\, x \in \widetilde{H}, x = ex = xe\ where\ e \in E(s)$, we have only to show that every $x \in \widetilde{H}$ has an inverse in $\widetilde{H}$.

If a semigroup S is regular, then every L-class of S contains at least one idempotent likewise the R-class.

If every element of S is contained in a subgroup of S, then every H-class of S also contains an idempotent.

In a class of semi-abundant semigroup, a completely regular semigroup S is called a regular band cryptogroup if the Green H-relation on S is regular band congruence. An abundant semigroup whose set of idempotent forms a regular band is called cyber group Guo and Shum so that the structure of regular cyber group was also first investigated by them.

Xiangrhi Kong, Yue Ding and K.P shum (2008) introduce the concepts of Green ~-relations on He-abundant semigroups. By using the generalized strong semilattice of semigroups, and show that $\widetilde{H} - cryptogroup$ is a regular $\widetilde{H} - cryptogroup$ if and only if it is an $\widetilde{HG}\ strong\ semilattice\ of$ completely $J - simple$ semigroup.

Kar-ping Shum, Lan Du and Yuq Guo 2010 gave some examples to illustrate some special properties of generalized Green's relations which are related to completely regular semigroup and abundant semigroup.

Preliminaries

Definition 1.1

Regular Semigroup-: This is a semigroup S in which every element is regular i.e for each a in S there exist x in S such that $a = axa$

Definition 1. 2

Simple semigroup-: A semigroup S without zero is said to be simple if S contain no proper two sided ideal.

Definition 1. 3

Semilattice -: A partially ordered set X is called a lower (upper) semilattice if , for all $a, b \in X_n$ the meet $a \wedge b$ (the join $a \vee b$) exists .If we have the stronger property that for every non-empty subset $Y\ of\ X$,the meet $\wedge \{y: y \in Y\}$ (the join $\vee \{y: y \in Y\}$) exist , then we say that X is a complete lower (upper) semilattice , if X is complete lower semilattice and a complete upper lattice we call it a complete lattice.

Definition 1.4

Complete Simple Semigroup:- A semigroup S (without zero) is complete if it is simple and contains a primitive idempotent or

A semigroup S is completely simple if

I. S is simple
II. To each a in S there exist idempotent e and f in S such that $ea = af = a$
III. Every idempotent of S is primitive i.e if $e \in E(s)$ by $f = ef = fe$ it follows $f = e$

Definition 1.5

Congruence**:-** let S be semigroup a relation $\rho\ on\ S$is called:

- Left compactible if $(\forall s, t, a \in S)(s, t) \in \rho \Longrightarrow (as, at) \in \rho$
- Right compactible if $(\forall s, t, a \in S)(s, t) \in \rho \Longrightarrow (sa, ta) \in \rho$
- Compactible if $(\forall s, t, x, y \in S)(s, t), (x, y) \in \rho \Longrightarrow (sx, ty) \in \rho$

A left (right) compactible equivalence is called a left (right) congruence. A compactible equivalence is called congruence.

Definition 1.6

Band: A semigroup in which every element is idempotent is called a Band

Definition 1.7

Normal band :- A band B is called normal if for $a, b, c \epsilon B\ cabc = cbac$

Definition 1.5

Clifford semigroup: A semigroup in which every element is a group element that is, lies in some subgroup. An element of a semigroup is a group element if and only if it is completely regular.

Every Clifford semigroup has a unique decomposition into groups, the classes of which are exactly the H-class, also Clifford semigroup is completely simple if it is simple.

The following conditions are equivalent for a Clifford semigroup

1) S is inverse
2) Every idempotent of S lies in the Centre, that is, it commutes with every element of S
3) Every one-sided of S is two sided
4) The Green relations H and D on S coincide
5) S is a semilattice of groups
6) S is subdirect products and group with zero.

Theorem1: A regular semigroup can be expressed as a union of group if and only if it is a semilattice of completely simple group.

Before the proof we will look at some lemmas:

Lemma 1: S is a semigroup that admits relative inverses, if it satisfies the following condition.

- To each elements a of S there exist an element e of S such that e is an idempotent element of a.

$$ea = ae = e$$

- a possesses an inverse a^1 relative to e in S

$$aa^1 = a^1a = e$$

Then e is idempotent
Proof:

$$e^2 = ea^1aa^1 = aea^1 = aa^1 = e$$

Lemma 2: if a is any element of S, and e is any element of S satisfying lemma 1, then e is idempotent.

Proof:

$$e^2 = aa^1aa^1 = aea^1 = aa^1 = e$$

Lemma 3: if a is any element of S, and if $e_1 and\ e_2$ both satisfy 1 then $e_1 = e_2$

Proof:
Let a_1 and a_2 be inverses of a relative to e_1 and e_2 respectively as given by (2)

$$aa_1 = a_1a = e_1, aa_2 = a_2a = e_2$$
$$then$$
$$e_1e_2 = e_1aa_2 = aa_2 = e_1$$
$$e_1e_2 = a_1ae_2 = a_1a = e_1$$
$$whence\ e_1 = e_2$$

A belong to the uniquely determined idempotent element e satisfying (1)

Lemma 4: The set S_e of all elements of S belong to the idempotent element e of S and is a group with identity e.

The set s_e of all elements of S belonging to the idempotent element e of S is a group with identity e.

Proof:

Let a and b be any two element of S, and let $a^1$and b^1 be inverses of a and b relative to e. Then e is evidently an identity element of ab and b^1a^1 is an inverse of ab relative to e.

$$abb^1a^1 = aea^1 = aa^1 = e$$
$$b^1a^1ab = b^1eb = b^1b = e$$

Hence $\in Se$. Evidently $e \in Se$, since $e^2 = e, and\ e$ is an identity element of Se . If a is any element of Se, a^1 an inverse of a relative to e, and $a_1b = ea^1e$

Then $eb = be = b$

$$ab = aea^1e = aa^1e = ee = e$$
$$a_1ba = ea^1ea = ea^1a = ee = e$$

Hence $b \in Se$ and b is an inverse of a therein, thus Se is a group

Now by (1) every element of S belongs to at least one of the groups Se, and by lemma 3 to exactly one, Hence S is the class sum of the mutually disjoint Se.

Note :

- Every inverse semigroup is a regular semigroup but the reverse does not follow

Lemma 5: if S is a semigroup admitting relative inverses and if e and f are idempotent of S such that $e \leq f$and f lies in SeS, then $e = f$

Proof

From $f \in xy$ for some x and y in S, Setting $a = fxf$, $b = fyf$ and using $fe = ef = e$ we have $aeb = fxffyf = fxey = fff = f$ together with $fa = af = a, fb = bf = b$

Let g be the idempotent to which a belongs. Then

$$f = aeb = gaeb = gf = a^{-1}af = a^{-1}a = g$$

Hence a belong to f, and similarly b belong to f. consequently

$$a^{-1}fb^{-1} = a^{-1}aebab^{-1} = fef = e$$

Since a^{-1} and b^{-1} are in Sf, this implies that e is in Sf where $e = f$

Lemma 6: A simple semigroup S with zero is completely simple iff it admits relative inverses.

Proof

Condition (1) for complete simple holds by hypothesis condition (ii) follows from lemma 3 condition (iii) holds from lemma 5 since $SeS = S$

Definition 1.7

Strong semilattice of a group;- let Y be a semilattice for each $\alpha \in Y$ let s_α be a semigroup and assume that $s_\alpha \wedge s_b = \emptyset$ if $\alpha \neq \beta$for each $\alpha, \beta \epsilon Y$ such that $\alpha \geq \beta$,there exists homomorphism.

$\varphi_{\alpha,\beta}: s_\alpha \longrightarrow s_{\alpha\beta}$ such that

- $\varphi_{\alpha,\alpha} = 1s_\alpha$
- For any $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$ so that $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}$ for multiplication on $S = \cup_{\alpha \in Y} S_\alpha$ in terms of multiplications in the s_α and the homomorphism $\varphi_{\alpha,\beta}$for each x in S_α and Y in S_β so that

  $$xy = x\varphi_{\alpha,\alpha\beta} y\varphi_{\beta,\alpha\beta}$$

  Then S with the multiplication above is a strong semilattice Y of semigroup $S_{\alpha,}$to be denoted by

  $$S[Y: S_{\alpha,}\varphi_{\alpha,\beta}]$$

  The homomorphism of S and if $S_\alpha \in \omega$ for all $\alpha \in Y$ and some class of semigroup ω then S is a strong semilattice of type ω

Definition 1.8

Completely regular semigroup:-This is a semigroup in which every element is in some subgroup of the semigroup.

Definition 1.9

Center of a semigroup:- The center of a semigroup $(S_{,\circ})$ denoted by $Z(s)$ is the subset of element in S that commute with every element in s symbolically: $z(s) = \{s \in S: \forall x \in S: sox = xos$

Theorem 2: If all idempotent of a completely regular semigroup S are central then the semigroup can be expressed as a strong semilattice of a group

To solve this let first state some lemmas

Lemma 6:

If an ideal a contains a single element of the group S_e then it contains all of S_e. In particular, if a^n belongs to an ideal a itself belongs to a

Proof:

Let a be an element in both a and S_e ,then a contains $a^{-1}a = a$ if b is any other element of S_e , then a contains $be = b$

Hence $a \supseteq Se$. The second statement then follows from the fact that a^n belongs to the same group $Se\ as\ a$

Back to the proof of theorem 2:

Let ω be any semilattice and to each α in ω assign a group S_α such that no two of them have an element in common.

To each pair of element $\alpha > \beta$ of ω assign a homomorphism $\varphi_{\alpha,\beta}$ of S_α into S_β such that if $\alpha > \beta > \gamma$ then

$$\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$$

Let $\varphi_{\alpha,\alpha}$ be the identical automorphism of S_α let S be the class sum of the group S_α, and define the product of any two elements $a_{\alpha,}$ and $b_\beta$of S ($a_\alpha\ in\ S_\alpha\ and\ b_\beta\ in\ S_\beta$) by

$$\gamma_\alpha b_\beta = (a_\alpha\varphi_{\alpha,\gamma})(b_\beta\varphi_{\beta,\gamma})$$

Where $\gamma = \alpha\beta$ is the product of $\alpha\ and\ \beta\ in\ \omega$

Conversely any semigroup S constructed in this fashion admits relative inverse and idempotent element of S_α in the center of S.

2 KG-System Of Semigroup

In this section, we introduce the concept of KG-strong semilattice decomposition of a semigroup S = (Y ; S_α), where Y is a semilattice and each S_α is a subsemigroup of S . In this paper, we always denote a semigroup S which is a semilattice Y of subsemigroups $S_\alpha(\alpha \in Y)$ by S = (Y ; S_α).

Definition 2.1 Let S = (Y ; S_α) be a semigroup. Suppose that the following conditions hold on the semigroup S.

I) ($\forall\alpha, \beta \in Y, \alpha \geq \beta$), there exists a family of homomorphisms

$\varphi_{d(\alpha,\beta)}$: $S_\alpha \longrightarrow S_\beta$, where d(α, β) ∈ D(α, β) and D(α, β) is a non-empty index set which runs between the interval [α, β]

II) (∀α ∈ Y), D(α, α) is a singleton set. Denote the element in D(α, α) by d(α, α). In this case, the homomorphisms $\varphi_{d(\alpha,\alpha)}$: $S_\alpha \longrightarrow S_\alpha$ is the identity automorphism of the semigroup S_α .

III) (∀α, β, γ ∈ Y,α ≥ β ≥ γ), if we write $\Phi_{\alpha,\beta} = \{\varphi_{d(\alpha,\beta)}: d(\alpha, \beta) \in D(\alpha, \beta)\}$ then $\Phi_{\alpha,\beta}\,\Phi_{\beta,\gamma} \subseteq \Phi_{\alpha,\gamma}$,
where $\Phi_{\alpha,\beta}\,\Phi_{\beta,\gamma} = \{\varphi_{d(\alpha,\beta)}\varphi_{d(\beta,\gamma)} \forall d(\alpha, \beta) \in D(\alpha, \beta), d(\beta, \gamma) \in D(\beta, \gamma)\}$

IV) $(\forall a \in S_\alpha)(\forall \beta \in Y)$, there exists a unique $\varphi^{a}_{d(\beta,\alpha\beta)} \in \Phi_{\beta,\alpha\beta}$ such that for any $b \in S_\beta$,

$$ab = (a\varphi^{b}_{d(\alpha,\alpha\beta)})(b\varphi^{a}_{d(\beta,\alpha\beta)}).$$

Remark : An index set is a set whose members or elements index (label) are members or elements of another set.

We call the above semilattice of semigroups S_α, with a set of structure homomorphisms $\Phi_{\alpha,\beta}$, a generalized strong semilattice if S = (Y ; S_α) satisfies the above conditions i)–iv).The above generalized strong semilattice of S_α is called the "G-strong semilattice" of S_α and is denoted by S = G[Y ; S_α, $\Phi_{\alpha,\beta}$]. The following definition is a stronger version of G-strong semilattices.

Definition 2.2

Let K be an equivalence relation on a G-strong semilattice of semigroups S = G[Y ; S_α, $\Phi_{\alpha,\beta}$]. We say that S is a KG-strong semilattice of semigroups if for each α, β ∈ Y , the mapping $a \longrightarrow \varphi^{a}_{d(\beta,\alpha\beta)}$ has the property that $\varphi^{a}_{d(\beta,\alpha\beta)} = \varphi^{b}_{d(\beta,\alpha\beta)}$ whenever the elements a, b ∈ S_α belong to the same K-class. Hence under the above multiplication, we obtain a G-strong semilattice of semigroups S determined by the equivalence relation K. We therefore call S a KG-strong semilattice of S_α and denote it by S = KG[Y; S_α, $\Phi_{\alpha,\beta}$]. Note. It is clear that the KG-strong semilattice is stronger than the G-strong semilattice but it is weaker than the strong semilattice. In fact, if ρ and δ are equivalences on the semigroup S = (Y ; S_α) with ρ ⊆ δ , then we observe that, δG[Y ; S_α, $\Phi_{\alpha,\beta}$] is "stronger" than ρG[Y ; S_α, $\Phi_{\alpha,\beta}$]. As the special case, 1_SG[Y ; S_α, $\Phi_{\alpha,\beta}$] is the "weakest" KG-strong semilattice since 1_S is the "smallest" equivalence relation on S and ηG[Y ; S_α, $\Phi_{\alpha,\beta}$] is the strongest KG-strong semilattice since η is the "biggest" equivalence relation on S , where 1_S is the identity equivalence relation on S and η is the semilattice congruence on S which divides S into S_α(α ∈ Y). Hence, it can be easily seen that ηG[Y ; S_α, $\Phi_{\alpha,\beta}$] is the usual strong semilattice since in this case, every index set D(α, β) is a singleton for α ≥ β on Y and hence there exists one and only one homomorphism in the set of structure homomorphisms $\Phi_{\alpha,\beta}$.

We observe that if the set D (α, β) for α ≥ β is just a singleton then the G-strong semilattices, KG-strong semilattices and strong semilattices are the same since there is no distinction between their respective multiplications.

Note: our k could be any one of the known Green's relations or any of the generalized Green relations, say L, R, D and $H, L^*, R^*, \widetilde{H}$ $e.t.c$

After the $Grean \sim relation$ $\tilde{L}, \tilde{R}, \widetilde{H}, \widetilde{D}$ and $\tilde{J}$ have been defined on a semigroup S, we will then consider the influence of the $Grean \sim relation$ for the purpose of clearity let us define some related terms

Definition 2.3

A left (right) ideal $L(R)$ of a semigroup is called a left (right)$\widetilde{ideal}$ of S if $\tilde{L}a \subseteq L(\tilde{R}a \subseteq R)$ holds, for all $a \in L(a \in R)$. We call a subset I of a semigroup S an $\widetilde{ideal}$ of S if it is both a left $\widetilde{ideal}$ and right $\widetilde{ideal}$.

From the above definition we can easily see that if the semigroup S is regular , then every left(right, two-sided) ideal of S is a left (right, two-sided) $\widetilde{ideal}$ secondly for any idempotent $e \in S$, the left (right) ideal $Se(Se)$ is always a left (right) $\widetilde{ideal}$ for if $a \in Se$, then it is clear that $a = ae$. And hence for every element $b \in \tilde{L}a$ we have $b = be \in Se$.

Definition 2.4

A semigroup S is called $\widetilde{H} - abundant$ if every $\widetilde{H} - class$ contains an idempotent of S. Clearly, the $\widetilde{H} - abundant$ semigroup in the class of semiabundant semigroup.

Definition 2.5

An $\widetilde{H} - abundant$ semigroup S is called completely $\tilde{J} - simple$ is $S \neq S^2$ and S does not contain any non-trivial proper $\widetilde{ideal}$ of S.

3 PROPERTIES OF $\widetilde{H} - abundant$

1) **Lemma 7:** let S be an $\widetilde{\boldsymbol{H}} - \boldsymbol{abundant}$ semigroup, the the following properties hold:
   - The $Green{\sim}relation$ $\breve{H}$ is a congruence on S iff $a, b \in S$ $(ab)^0 = (a^0b^0)^0$
   - If e, f are are $\breve{D}$ $related$ $idempotents$ of S, then eDf
   - $\breve{D} = \tilde{L}o\tilde{R} = \tilde{R}o\tilde{L}$
   - Then If e, f are idempotents in S such that eJf then eDf

Proof.

(i) We need to show that $\widetilde{H}$ is compatible with the semigroup multiplication of S since $\widetilde{H}$ is an equivalent relation on S. Let $(a, b) \in \widetilde{H}$ and $c \in S$. Then $(ca)^0 = (c^0a^0)^0 = (c^0b^0)^0$=$(cb)^0$ and hence, $\widetilde{H}$ is left compatible to the semigroup mulcatiplication. In the same way, $\widetilde{H}$ is right compatible with the semigroup multiplication hence, is $\widetilde{H}$ a congruence on S.

(ii) Since $e\widetilde{D}f$, there exist elements $a_1,\cdots,a_1$ of S such that $e\tilde{L}a_1\tilde{R}a_2\cdots a_k\tilde{L}f$. Since S is an $\widetilde{H}$-abundant semigroup, $eLa_1^0Ra_2^0\ \cdots\ a_k^0Lf$. Thus eDf.

(iii)

Proof

$$\tilde{L} = \{(a,b) \in S \times S\colon (\forall e \in E(s))ae = a \Leftrightarrow be = b$$

$$\tilde{R} = \{(a,b) \in S \times S\colon (\forall e \in E(s))ea = a \Leftrightarrow eb = b$$

Let S be a semigroup, let $a,b \in \tilde{R} \circ \tilde{L}$ then there exists $C\ in\ S$

Such that $a\tilde{L}$c and $c\tilde{R}b$ and $c\tilde{L}$b so that

$ae = c \quad fc = b$
$ce = a \quad fb = c$

Were $e,f \in E(S)$

Then let $u = fce$

$$fa = fce = u$$

$$fu = ffce = fbe = ce = a$$

So that we have $a\tilde{R}u$ secondly

$$ue = fcee = fae = fc = b$$

$be = fce = u$ so that we will have $u\tilde{L}b$
Hence $a,b \in \tilde{L}0\tilde{R}$

(iv) Since $SeS = SfS$, there exist elements x,y,s,t in S such that $f = set$ and $e = xfy$. Let $h = (fy)^0$ and $k = (se)^0$. Then $hfy = fy = ffy$ and so $h = h^2 = fh$ and $sek = se = see$, and thereby, $k = k^3 = ke$. Hence, hf, ek are the idempotents satisfying the relations $hfRh$ and Lk . These imply that $ehfReh$ and $ekfLkf$. Now by $eh = xfyh = xfy = e\ and\ kf = kset = set = f$, we have $eRefLf$. This shows that eDf.

2) If a semigroup is $\widetilde{H} - abundant\ semigroup$ then every element of S is abundant, meaning that it has a dense set of idempotents in it's $\widetilde{H} - class\ semigroup$.
3) If a semigroup S is $\widetilde{H} - abundant$, then S is regular, meaning that every element has an inverse.

Lemma 8: If a is a regular element of a semigroup S, say $axa = a$ with $x\ in\ S$ then a has at least one inverse in S, in particular xax

Proof :

Let $b = xax$ then

$$aba = axaxa = ax(axa) = axa = a$$

$$bab = xaxaxax = x(axa)xax = xaxax = xax = b$$

Since $\widetilde{H}$ has been proven to be regular invariably every element in $\widetilde{H}$ is a regular element and from lemma 8 every regular element has an inverse

Therefore if every element has an inverse then $\widetilde{H}$ has an inverse.

It won't be wrong to say $\widetilde{H} - abundant\ semigroup$ is an inverse semigroup.

4) If a semigroup S is $\widetilde{H} - abundant$, then S is an inverse semigroup, i.e every element has a unique inverse
5) The $\widetilde{H} - abundant$ is a special case of Green's L- and R-relations and thus, many results about Green's relations can be applied to $\widetilde{H} - abundant\ semigroup$
6) The union of $\widetilde{H} - abundant\ semigroup$ is also $\widetilde{H} - abundant\ semigroup$

Examples of $\widetilde{H} - abundant\ semigroup$

A 2x2 matrix can be used to illustrate $\widetilde{H} - abundant\ semigroup$.

Let consider 2x2 matices with integer entries, denoted by:

$$\rho = \{(a, b, c, d)/a, b, c, d\ \in Z\}$$

With the usual multiplication operation, $\rho\ forms\ a\ semigroup$. For ρ to satisfy the $Green{\sim}\ relation$ we need to define a specific operation and identify the required element.

Lets define the operation $*\ as\ follows$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} \max(a, e) & \max(b, f) \\ \max(c, g) & \max(d, h) \end{pmatrix}$$

The operation combines matrixes by taking the maximum value for each entry.

Remark:

- The identity matrix with respect to the operation defined above is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

- The idempotent matrix will be all 2x2 matrices of the form $\begin{pmatrix} a & a \\ b & b \end{pmatrix}$

3. RELATING GROUP HOMORPHISM TO SEMIGROUP HOMOMORPHISM

Firstly, we will look at similarities in constructing a homomorphism for both a semigroup and a group admitting semilattice decomposition.

Let's look at similarities in the steps:

1) Identify the decomposition: recognize the semilattice decomposition in both cases
2) Define the target structure: specify the target semigroup or group
3) Map elements: Define the homomorphism by mapping elements from each semilattice to the target structure
4) Preserve operation: ensure the homomorphism preserves the operation with each semilattice and between elements from different semilattices
5) Check that the homomorphism is well define

EXAMPLE

Let define a semigroup

$S = \{a, b, c, d\}$ with operation $*$ define on it such that:

$$a * b = c, b \circledast c = a, c \circledast a = b, a \circledast a = b \circledast b = c \circledast c = d,$$

$$d \circledast x = x \circledast d = d \; for \; all \; x \; in \; S$$

S admits semilattice decomposition decomposition:

$S = S_1 \sqcup S_2$ where $S_1 = \{a, b. c\}$ and $S_2 = d$

Identity automorphism on S

$$\rho: S_n \to S_n$$

Define by $\rho(x) = x \; for \; all \; x \; in \; S$

For groups we have $G = \{e, a, b, c, d\}$ with operation $\odot$ defined by

$e \odot x = x \odot e = x, \; a \odot b = c, \; b \odot c = a, \; c \odot a = b, \; a \odot a = b \odot b = c \odot c = d$

$$d \odot x = x \odot d = d(\; absorbing \; element)$$

G admits semilattice decomposition so that $G = \; G_1 \sqcup G_2$ where $G_1 = \{e, a, b, c, \}$ and $G_2\{d\}$

Identity automorphism on G:

$$\rho: G \to G \; define \; by \; \rho(x) = x \; for \; all \; x \; in \; G$$

Let's outline the similarities from the example above:

1) Both S and G have semilattice decomposition into two componenets
2) The identity automorphism ρ maps every element to itself in both S and G
3) The operators $\circledast, \odot$ are preserved by ρ in both cases
4) The component S_1 and G_1 have similar structures with $\{a, b. c\}$ forming a cyclic semigroup in both cases

RELATING GROUP THEORY TO $\widetilde{H} - ABUNDANT\ SEMIGROUP$

$\widetilde{H} - abundant$ Semigroup is a type of semigroup that satisfies certain properties. Here's how group theory relate to it:

- Subsemigroups: in group theory, subgroups are subsets that are closed under the operation. similarly, in a $\widetilde{H} - abundant$ semigroup, subsemigroup can be identified, which are subsets closed under the operation

Example: consider the semigroup S=$\{a, b, c\}$ $with\ operation\ \circ$ define by $a \circ b = c, b \circ c = a, c \circ a = b$

The subset $\{a, b\}$ is a subsemigroup

- Homomorphism: Group homomorphisms preserve the operation, similarly, semigroup homomorphisms between $\widetilde{H} - abundant$ semigroups preserve the operation.

Example: consider two semigroup S=$\{a, b, c\}$ $and\ T = \{x, y, z\}$ with operation $\circ\ and\ \circledast$ respectively. A homomorphism $\rho: S \longrightarrow T$ can be defined by:

$$\rho(a) = x, \rho(b) = y, \rho(c) = z \text{ Such that}$$

$$\rho(a \circ b) = \rho(c) = z = x \circledast y = \rho(a) \circledast \rho(b)$$

- Congruences: in group theory, congruences are equivalence relations that preserve the operation. Similarly, in a $\widetilde{H} - abundant$ semigroup congruences can be defined, which are equivalence relations that preserve the operation.

Example: consider the semigroup S=$\{a, b, c\}$ with operation$\circ$. A congruence relation $\equiv$ can be defined by:

$$a \equiv b, b \equiv c, c \equiv a\ such\ that\ if\ a \equiv b, then\ a \circ c \equiv b \circ c$$

These are just a few examples of how group theory concepts relate to $\widetilde{H} - abundant$ semigroups. The connections can be further explored in terms of ideals, Green's relations, and other semigroup properties.

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